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ESTIMATIONS OF THE PARAMETERS OF THE GENERALIZED QUADRATIC HAZARD RATE DISTRIBUTION USING TYPE-II CENSORED DATA

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Abstract

Recently, Sarhan [8] introduced a new distribution named generalized quadratic hazard rate distribution. In this paper, we deal with the problem of estimating the parameters of this distribution based on Type II censored data. The maximum likelihood and least square techniques are used. For illustrative purpose, the results obtained are applied on sets of real data.

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1. Introduction

Recently, many new distributions, generalizing well-known distributions used to study lifetime data, have been introduced. Mudholkar and Srivastava [6] presented a generalization of the Weibull distribution called the exponentiated(generalized)-Weibull distribution, GWD. They showed that this generalization not only includes distributions with bathtub and unimodal hazard rates but provides a broader class of monotone hazard rates. The generalized exponential distribution, GED, introduced by Gupta and Kundu [3]. Nadarajah and Kotz [7] introduced four exponentiated type distributions: the exponentiated gamma, exponentiated Weibull, exponentiated Gumbel, and the exponentiated Frchet distribution. They provided a treatment of the mathematical properties for each distribution. Sarhan and Kundu [9] presented a generalization of the linear hazard rate distribution called the generalized linear hazard rate distribution, GLFRD. They explained that this distribution can have increasing, decreasing and bathtub shaped hazard rate functions which are quite desirable for data analysis purposes. Sarhan et al. [10] obtained Bayes and maximum likelihood estimates of the three parameters of the generalized linear hazard rate distribution based on grouped and censored data. Recently, Sarhan [8] introduced a generalization of the quadratic hazard rate distribution called the generalized quadratic hazard rate distribution (GQHRD).

We intend, in this paper, to estimate the unknown parameters of the GQHRD based Type-II censored data.

The generalized quadratic hazard rate distribution generalizes several well known distributions. Among these distributions are the quadratic hazard rate, the linear failure rate, the generalized linear failure rate, the generalized exponential and the generalized Rayleigh distributions.

The GQHRD may have an increasing (decreasing) hazard function or a bathtub shaped hazard function or an upside-down bathtub shaped hazard function. This property enables this distribution to be used in many applications in several areas, such as reliability, life testing, survival analysis and others.

The traditional quadratic hazard rate distribution with three parameters $a, b, c,$ denoted as $\text{QHRD}(a, b, c),$ has the following cdf:

$$F(x;\, a,\, b,\, c) = 1 - \exp\left\{-\, ax - \frac{b}{2}x^2 - \frac{c}{3}x^3\right\},\; x \geq 0, \tag{1.1}$$

where $a \geq 0, c \geq 0$ and $b \geq -2\sqrt{ac}.$ This restriction on the parameter space is made to be insure that the hazard function with the following form is positive, see Bain [1],

$$h(x;\, a,\, b,\, c) = a + bx + cx^2, \quad x \geq 0. \tag{1.2}$$

The $\text{QHRD}\,(a, b, c)$ generalizes exponential, Rayleigh, Weibull with shape parameter equals 3 and linear hazard rate, see for example Bain [1]. Obviously, the exponential distribution (say $\text{ED}(a)$) can be obtained from $\text{QHRD}\,(a, b, c)$ when $b = 0, c = 0,$ the Rayleigh distribution (say $\text{RD}(b)$) can be derived from $\text{QHRD}\,(a, b, c)$ when $a = 0, c = 0,$ the Weibull with shape parameter equals 3 (say $\text{WD}(c, 3)$) can be derived from $\text{QHRD}\,(a, b, c)$ when $a = 0, b = 0$ and linear hazard rate distribution (say $\text{LFRD}\,(a, b)$) can be derived from $\text{QHRD}\,(a, b, c)$ when $c = 0.$

Sarhan [8] introduced the generalized quadratic hazard rate distribution with four parameters a, b, c, d and denoted by GQHRD $(a, b, c, d).$ The cdf of $\text{GQHRD}\,(a, b, c, d)$ takes the form,

$$F(x;\, a,\, b,\, c,\, d) = \left[1 - e^{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)}\right]^d,\; x \geq 0, \tag{1.3}$$

where $a \geq 0, c \geq 0, d > 0$ and $b \geq -2\sqrt{ac}.$

It is important to mention here that when d is a positive integer, the cdf of $\text{GQHRD}\,(a, b, c, d)$ represents the cdf of the maximum of a simple random sample of size d from the $\text{QHRD}\,(a, b, c).$

This distribution generalizes the following distributions: the $\text{GLFRD}\,(a, b, c)$ when $c = 0;$ the $\text{GED}\,(a, d)$ when $b = 0, c = 0, a > 0;$

the GRD (b, d) when $a = 0, c = 0, b > 0$; and QHRD (a, b, c) when $d = 1$.

The main object of this article is to estimate the four unknown parameters of the GQHRD (a, b, c, d). We use the maximum likelihood and least squares procedures to derive such estimates. The estimators are obtained by using the data of type II censoring testing without replacement. Also the asymptotic confidence intervals of the parameters are discussed. Further, we study whether this distribution fits a set of real data better than other distributions. Two criteria are used for this purpose. These are the Kolmogorov-Smirnov test statistic and the values of the log-likelihood function. Monte Carlo simulation technique is used to study the performance of the estimators obtained. For this purpose, we used the mathematical program MATLAB 7.

The rest of this paper is organized as follows. Some properties of the GQHRD (a, b, c, d) are presented in Section 2. Section 3 presents the model assumptions and notations. Section 4 gives the parameter estimations using both maximum likelihood and least squares techniques. We use a set of real data in Section 5 as an application.

2. The GQHRD

The survival function of the GQHRD (a, b, c, d) takes the following form

$$S(t) = 1 - \left[1 - e^{-(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3)}\right]^d, \quad t \geq 0. \tag{2.1}$$

The pdf of the GLFRD (α, β, γ) is the probability density function, pdf of GQHRD (a, b, c, d) takes the following form

$$f(x; a, b, c, d) = d(a + bx + cx^2)\left[1 - e^{-(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3)}\right]^{d-1} e^{-(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3)}, \; x \geq 0. \tag{2.2}$$

Figure 1 shows some patterns of the pdf of GQHRD (a, b, c, d), which may have a single mode or no mode at all.

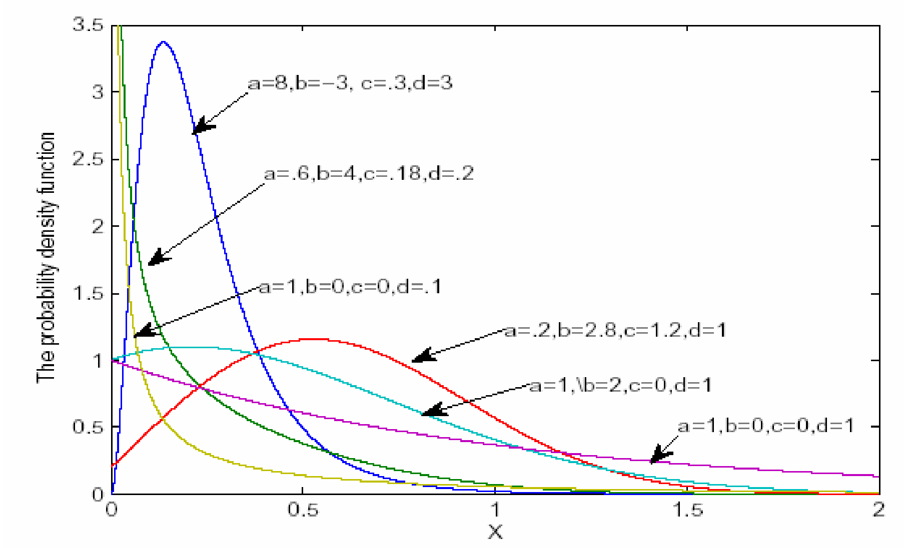

Figure 1. Different patterns of the probability density function.

The hazard rate function of GQHRD (a, b, c, d) is

$$h(x; a, b, c, d) = \frac{d(a + bx + cx^2)\left[1 - e^{-(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3)}\right]^{d-1} e^{-(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3)}}{1 - \left[1 - e^{-(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3)}\right]^{d}}. \quad (2.3)$$

The hazard rate function is such that:

• if $d = 1$, the hazard function is either increasing (if $b > 0$) or constant (if $b = 0$ and $a > 0$);

• when $d > 1$, the hazard function should be: (1) increasing if $b > 0$; (2) upside-down bath-tub shaped if $b < 0$; and

• if $d < 1$, then the hazard function will be: (1) decreasing if $b = 0$ or (2) bath-tub shaped if $b \neq 0$.

Figure 2. shows different patterns of the hazard rate function.

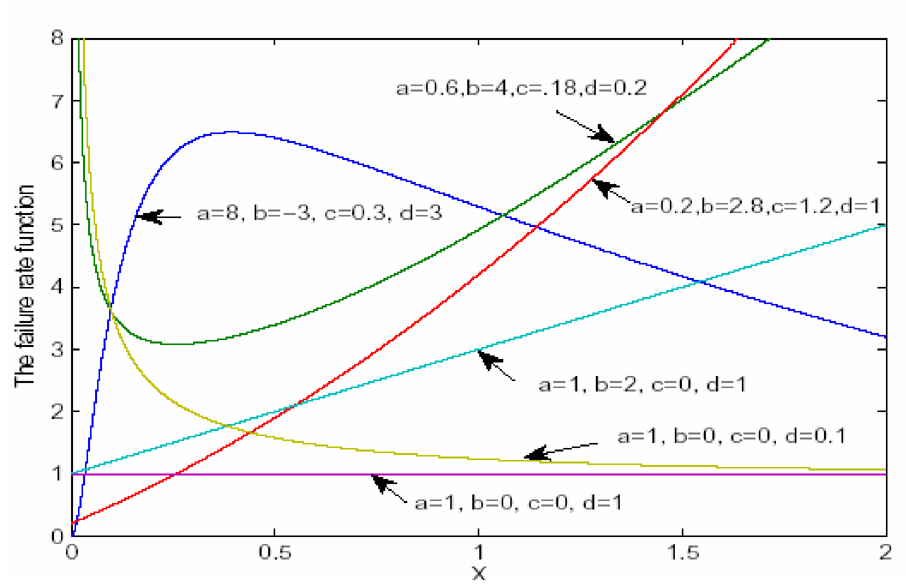

Figure 2. Different patterns of the hazard rate function.

3. Parameters Estimation

In this section, we use the maximum likelihood and least squares procedures to derive point estimates of the unknown parameters of the GQHRD(a, b, c, d). Also, we will derive the asymptotic interval estimates of the parameters.

Henceforth we shall consider the data of type II censoring testing without replacement. In such type of data, it is assumed that n identical items are put on the life test. The testing process is terminated at the time of rth item failure. The number of observations r is decided before the data are collected. Assume the r times to failures are $x_1, x_2, \ldots, x_r$. Let $\underline{x}$ denote the information obtained from the life testing. It means the number of all items to be tested n, the number of failed items r, and their times to failure. That is $\underline{x} = \{n, r; x_1, x_2, \ldots, x_r\}$. It is assumed also that the life time of each item follows the GQHRD(a, b, c, d) with cdf given by (1.3).

3.1. The maximum likelihood estimators

In this subsection, we use the maximum likelihood procedure to derive the point and interval estimates of the parameters.

3.1.1. Point estimators

The likelihood function of $\underline{x}$ is, see Lawless [4],

$$L(\underline{x}) = \{S(x_r;\, a,\, b,\, c,\, d)\}^{(n-r)}\left\{\prod_{i=1}^{r} f(x_i;\, a,\, b,\, c,\, d)\right\}. \tag{3.1}$$

Substituting (2.1) and (2.2) into (3.1), we get

$$L(\underline{x}) = d^r \left\{\prod_{i=1}^{r} (a + bx_i + cx_i^2)\left[1 - e^{-(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3)}\right]^{d-1}\right\} e^{-\sum_{i=1}^{r}(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3)}$$

$$\left\{1 - \left[1 - e^{-(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3)}\right]^{d}\right\}^{(n-r)}. \tag{3.2}$$

The log-likelihood, denoted $\mathcal{L}(\underline{x})$, is

$$\mathcal{L}(\underline{x}) = r \ln d - a\sum_{\mathrm{i}=1}^{\mathrm{r}} x_i - \frac{b}{2}\sum_{i=1}^{r} x_i^2 - \frac{c}{3}\sum_{i=1}^{r} x_i^3 + \sum_{i=1}^{r} \mathrm{In}\,(a + bx_i + cx_i^2)$$

$$+ (d-1)\sum_{i=1}^{r} \ln\left[1 - e^{-(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3}\right]$$

$$+ (n-r)\ln\left\{1 - \left[1 - e^{-(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3}\right]^{d}\right\}. \tag{3.3}$$

Calculating the partial derivatives of $\mathcal{L}$ with respect to a, b, c, d and equating each to zero, we can get the likelihood equations as in the following system of nonlinear equations of a, b, c and d.

$$0 = \frac{r}{d} + \sum_{i=1}^{r} \ln\left[1 - e^{-(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3)}\right] + (n-r)\sum_{i=1}^{r} \frac{\ln\left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]}{1 - \left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]^{-d}},$$

$$0 = -\sum_{i=1}^{r} x_i + \sum_{i=1}^{r} \frac{1}{a + bx_i + cx_i^2} + (d-1)\sum_{i=1}^{r} \frac{x_i}{e^{(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3)} - 1}$$

$$- (n-r) x_r d \frac{e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)} \left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]^{d-1}}{1 - \left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]^d},$$

$$0 = -\frac{1}{2}\sum_{i=1}^{r} x_i^2 + \sum_{i=1}^{r} \frac{x_i}{a + bx_i + cx_i^2} + \frac{1}{2}(d-1)\sum_{i=1}^{r} \frac{x_i^2}{e^{(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3)} - 1}$$

$$- \frac{1}{2}(n-r) x_r^2 d \frac{e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)} \left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]^{d-1}}{1 - \left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]^d},$$

$$0 = -\frac{1}{3}\sum_{i=1}^{r} x_i^3 + \sum_{i=1}^{r} \frac{x_i^2}{a + bx_i + cx_i^2} + \frac{1}{3}(d-1)\sum_{i=1}^{r} \frac{x_i^3}{e^{(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3)} - 1}$$

$$- \frac{1}{3}(n-r) x_r^3 d \frac{e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)} \left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]^{d-1}}{1 - \left[1 - e^{-(ax_r + \frac{b}{2}x_r^2 + \frac{c}{3}x_r^3)}\right]^d}. \tag{3.4}$$

To find out the maximum likelihood estimators of a, b, c and d, we have to solve the above system of nonlinear equations (3.4) with respect to a, b, c and d. As it seems, this system has no closed form solution in a, b, c and d. Then we have to use a numerical technique method, such as Newton-Raphson method, to obtain the solution.

3.1.2. Asymptotic confidence bounds

Since the MLE of the parameters cannot be derived in closed forms, we cannot get the exact confidence bounds of the parameters. The idea is to use the large sample approximation. The maximum likelihood estimators of $\theta = (a, b, c, d)$ can be treated as being approximately multi-normal with mean θ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$(\hat{\theta} - \theta) \rightarrow N_4(0, \mathbf{I}^{-1}(\hat{\theta})), \tag{3.5}$$

where $\mathbf{I}^{-1}(\hat{\theta})$ is the variance-covariance matrix of the unknown parameters θ. The element $I_{ij}(\hat{\theta}),\ i,\ j = 1, 2, 3, 4,$ of the 4×4 matrix $\mathbf{I}^{-1}$ is given by

$$I_{ij}(\hat{\theta}) = -\mathcal{L}_{\theta_i\theta_j}\,|_{0=\hat{\theta}}. \tag{3.6}$$

Using the first derivatives of $\mathcal{L}$, the second partial derivatives of the log-likelihood function are derived in the Appendix.

Therefore, the approximate $100(1-\vartheta)\%$ two-sided confidence intervals for a, b, c, d are, respectively, given by

$$\hat{a} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{11}^{-1}(\hat{\theta})},\quad \hat{b} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{22}^{-1}(\hat{\theta})},\quad \hat{c} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{33}^{-1}(\hat{\theta})},\quad \hat{d} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{44}^{-1}(\hat{\theta})}.$$

Here, $Z_{\vartheta/2}$ is the upper $(\vartheta/2)$th percentile of the standard normal distribution.

3.2. The least squares procedure

In this subsection, we shall derive the least square estimators (LSEs) of the four parameters a, b, c, d. Given the observed lifetimes $x_1 < x_2 < \ldots < x_m$ in a certain censored sample from the GQHRD (a, b, c, d). Then the least square estimates of the parameters $a, b, c, d,$ denoted $\hat{a}_R, \hat{b}_R, \hat{c}_R, \hat{d}_R$ respectively, can be obtained by minimizing the following quantity with respect to a, b, c, d

$$Q = \sum_{i=1}^{m}\left\{\hat{F}_i - \left[1 - e^{-ax_i - \frac{b}{2}x_i^2 - \frac{c}{3}x_i^3}\right]^d\right\}^2, \tag{3.7}$$

where $\hat{F}_i = \hat{F}(x_i)$ is the empirical estimate of $F(x)$ at the observation $x_i,\ i = 1, 2, \ldots, m.$

That is to get $\hat{a}_R, \hat{b}_R, \hat{c}_R, \hat{d}_R,$ we have to solve the following system of non-linear equations with respect to $a, b, c, d.$

The partial derivatives of Q with respect to a, b, c, d are respectively

$$\frac{\partial Q}{\partial \theta_j} = -2d\sum_{i=1}^{m}\left\{\hat{F}_i - \left[1 - e^{-ax_i - \frac{b}{2}x_i^2 - \frac{c}{3}x_i^3}\right]^d\right\}F_{Q,\ \theta_j}(x_i),\ j = 1, 2, 3,$$

$$\frac{\partial Q}{\partial d} = -2\sum_{i=1}^{m}\left\{\hat{F}_i - \left[1 - e^{-ax_i - \frac{b}{2}x_i^2 - \frac{c}{3}x_i^3}\right]^d\right\}[F_Q(x_i)]^d \ln F_Q(x_i), \tag{3.8}$$

where $\theta_1 = a, \theta_2 = b, \theta_3 = c$ and

$$F_Q(x_i) = 1 - e^{-ax_i - \frac{b}{2}x_i^2 - \frac{c}{3}x_i^3},$$

$$F_{Q,\ \theta_j}(x_i) = \frac{1}{j}x_i^j e^{-ax_i - \frac{b}{2}x_i^2 - \frac{c}{3}x_i^3}.$$

Setting $\frac{\partial Q}{\partial a} = 0, \frac{\partial Q}{\partial b} = \frac{\partial Q}{\partial c} = \frac{\partial Q}{\partial d} = 0,$ we shall get the following system of non-linear equations

$$0 = \sum_{i=1}^{m}\left\{\hat{F}_i - \left[1 - e^{-ax_i - \frac{b}{2}x_i^2 - \frac{c}{3}x_i^3}\right]^d\right\}F_{Q,\ \theta_j}(x_i),\ j = 1, 2, 3,$$

$$0 = \sum_{i=1}^{m}\left\{\hat{F}_i - \left[1 - e^{-ax_i - \frac{b}{2}x_i^2 - \frac{c}{3}x_i^3}\right]^d\right\}[F_Q(x_i)]^d \ln F_Q(x_i). \tag{3.9}$$

Solving the above system with respect to $a, b, c, d,$ we can get the LSEs $\hat{a}_R, \hat{b}_R, \hat{c}_R, \hat{d}_R$. As it seems the above system has no explicit solution. Therefore, we have to use a numerical technique to get the solution.

4. Illustrative Examples

In this section we present practical applications of the theoretical results discussed in the preceding sections with two examples. One example involves a large sample and the other with a small sample.

4.1. Example 1

This example is from McCool [5] giving the fatigue life in hours of ten bearing of a certain type. These data are as follows:

152.7, 172.0, 172.5, 173.3, 193.0, 204.7, 216.5, 234.9, 262.6, 422.6

In this case, if we assume a Type-II censored sample of size $m = 8$, we obtain the maximum likelihood estimates of the parameters of the following distributions: (1) E(a), (2) R(b), (3) QHRD(a, b, c), (4) GE(a, b) (5) GR(a, b), (6) GLFRD(a, b, c) and (7) GQHRD(a, b, c, d). Also, we compared these four distributions to fit the data. For comparison purpose, we use: (1) the likelihood ratio test statistics and the corresponding p-value, and (2) the mean square of the difference between the empirical cdf and fitted cdf, say MSD, using each model. Not that MSD is computed by the following relation

$$\text{MSD} = \frac{1}{m}\sum_{i=1}^{m}\left(\hat{F}_i - FE_i\right)^2,$$

where $\hat{F}_i$ and FE_i are the empirical and the estimated cdf computed at x_i. The estimated cdf is computed by replacing the parameters of the model adopted with their the MLE. Table 1, gives the results obtained.

Table 1. The results for example 1

Dist.	Parameter estimates	$\mathcal{L}$	Λ	p-value	MSD
E	2.0213×10^{-3}	-52.129	20.691	1.221×10^{-4}	0.04021
R	3.958×10^{-5}	-47.195	10.823	0.013	0.02306
QHRD	$4.042\times10^{-3}, -7.534\times10^{-5}, 5.741\times10^{-7}$	-45.536	7.505	6.152×10^{-3}	0.01678
GE	6.649×10^{-3}, 2.205	-48.973	14.378	7.550×10^{-4}	0.02684
GR	6.597×10^{-5}, 2.193	-44.466	5.364	0.068	0.01064
GLFRD	0.001, 7.316×10^{-5}, 3.278	-43.764	3.961	0.047	8.222×10^{-3}
GQHRD	$8.318\times10^{-8}, -5.451\times10^{-5}, 1.035\times10^{-6}$, 4.221	-41.784	-	-	9.635×10^{-3}

The results shown in Table 1 imply that the GQHRD fits the given data better than all other distributions mentioned above.

4.2. Example 2

The data set is given by Birnbaum and Saunders [2] on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31000 psi. The data are presented in Table 2.

Table 2. Fatigue lifetime data presented by Birnbaum and Saunders [2]

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

Assuming Type-II censored samples with different sizes $m = 101, 100,$ $95,$ $90,$ $80,$ $70,$ $60,$ and $50,$ we obtain (1) the maximum likelihood estimates of the parameters of the $\text{GQHRD}(a, b, c, d)$; (2) the standard deviation of the parameter estimates; and (3) the 95% confidence intervals of the parameters. Tables 3-5 gives the results obtained.

Table 3. MLE of the parameters a, b, c, d for example 2

r	a	b	c	d
101	2.023×10^{-3}	-5.581×10^{-5}	3.009×10^{-6}	5.592
100	6.325×10^{-3}	-1.081×10^{-4}	3.329×10^{-6}	8.054
95	6.1×10^{-3}	-1.094×10^{-4}	3.393×10^{-6}	8.059
90	6.097×10^{-3}	-1.081×10^{-4}	3.382×10^{-6}	8.051
80	7.769×10^{-3}	-1.918×10^{-4}	3.907×10^{-6}	7.186
70	8.001×10^{-3}	-2.578×10^{-4}	4.462×10^{-6}	6.146
60	7.177×10^{-3}	-1.217×10^{-4}	3.352×10^{-6}	8.275
50	7.175×10^{-3}	-1.197×10^{-4}	3.381×10^{-6}	8.522

Table 4. Standard deviation of the MLE of parameters a, b, c, d for example 2

r	a	b	c	d
101	9.386×10^{-3}	3.549×10^{-4}	2.324×10^{-6}	6.658
100	0.017	3.357×10^{-4}	2.153×10^{-6}	11.801
95	0.015	3.610×10^{-4}	2.352×10^{-6}	10.794
90	0.014	3.801×10^{-4}	2.456×10^{-6}	8.829
80	0.013	4.127×10^{-4}	2.654×10^{-6}	5.231
70	0.013	4.347×10^{-4}	2.863×10^{-6}	3.479
60	0.020	5.092×10^{-4}	3.462×10^{-6}	12.423
50	0.021	5.712×10^{-4}	3.984×10^{-6}	13.34

Table 5. 95% Confidence intervals of the parameters a, b, c, d for example 2

r	a	b	c	d
101	(0, 0.020)	$(-7.514 \times 10^{-4}, 6.398 \times 10^{-4})$	$(0, 7.564 \times 10^{-6})$	(0, 18.642)
100	(0, 0.039)	$(-7.660 \times 10^{-4}, 5.497 \times 10^{-4})$	$(0, 7.548 \times 10^{-6})$	(0, 31.184)
95	(0, 0.036)	$(-8.169 \times 10^{-4}, 5.981 \times 10^{-4})$	$(0, 8.004 \times 10^{-6})$	(0, 29.215)
90	(0, 0.034)	$(-8.532 \times 10^{-4}, 6.370 \times 10^{-4})$	$(0, 8.195 \times 10^{-6})$	(0, 25.355)
80	(0, 0.034)	$(-1.001 \times 10^{-3}, 6.171 \times 10^{-4})$	$(0, 9.109 \times 10^{-6})$	(0, 17.437)
70	(0, 0.033)	$(-1.110 \times 10^{-3}, 5.943 \times 10^{-4})$	$(0, 1.007 \times 10^{-5})$	(0, 12.965)
60	(0, 0.046)	$(-1.120 \times 10^{-3}, 8.764 \times 10^{-4})$	$(0, 1.014 \times 10^{-5})$	(0, 32.623)
50	(0, 0.048)	$(-1.239 \times 10^{-3}, 9.999 \times 10^{-4})$	$(0, 1.119 \times 10^{-5})$	(0, 34.668)

5. Conclusion

In this paper we discussed the parameter estimation of the GQHRD (a, b, c, d) based on Type-II censored data. The maximum likelihood and least square techniques have been used. The GQHRD (a, b, c, d) is tested against different distributions using a set of real data. Based on the two criteria (the values of the log-likelihood function and average K-S test statistics), we found that the GQHRD (a, b, c, d) fits the data better than those compared distributions. Further, we used another real data set with a large size to derived the asymptotic confidence intervals of the parameters with different censoring sizes.

Appendix

The second partial derivatives of the log-likelihood function can be derived as in the following forms.

$$\frac{\partial^2 \mathcal{L}}{\partial d^2} = \frac{-m}{d^2} - \frac{(n-m)\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d \left[\ln\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]\right]^2}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial d \partial a} = \sum_{i=1}^{m} \frac{x_i}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]}$$

$$-\frac{(n-m)dx_m e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-1}\ln\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2}$$

$$+\frac{(n-m)dx_m e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}\ln\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$+\frac{(n-m)x_m e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d},$$

$$\frac{\partial^2\mathcal{L}}{\partial d\partial b}=\frac{1}{2}\sum_{i=1}^{m}\frac{x_i^2}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]}$$

$$-\frac{1}{2}\frac{(n-m)dx^2{}_m e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-1}\ln\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2}$$

$$+\frac{1}{2}\frac{(n-m)dx^2{}_m e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}\ln\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$+\frac{1}{2}\frac{(n-m)x_m^2 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$\frac{\partial^2\mathcal{L}}{\partial d\partial c}=\frac{1}{3}\sum_{i=1}^{m}\frac{x_i^3}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]}$$

$$-\frac{1}{3}\frac{(n-m)dx_m^3 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-1}\ln\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2}$$

$$+\frac{1}{3}\frac{(n-m)dx_m^3 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}\ln\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$+\frac{1}{3}\frac{(n-m)x_m^3 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d},$$

$$\frac{\partial^2\mathcal{L}}{\partial a^2}=-\sum_{i=1}^{m}\frac{1}{(a+bx_i+cx_i^2)^2}-(d-1)\sum_{i=1}^{m}\frac{x_i^2 e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]^2}$$

$$-\frac{(n-m)d^2x_m^2 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-2}}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2}$$

$$+\frac{(n-m)d(d-1)x_m^2 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-2}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$-\frac{(n-m)dx_m^2 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d},$$

$$\frac{\partial^2\mathcal{L}}{\partial a\partial b}=-\sum_{i=1}^{m}\frac{x_i}{(a+bx_i+cx_i^2)^2}-\frac{1}{2}(d-1)\sum_{i=1}^{m}\frac{x_i^3 e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]^2}$$

$$-\frac{1}{2}\frac{(n-m)d^2x_m^3e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-2}}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2}$$

$$+\frac{1}{2}\frac{(n-m)d(d-1)x_m^3e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-2}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$-\frac{1}{2}\frac{(n-m)dx_m^3e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d},$$

$$\frac{\partial^2\mathcal{L}}{\partial a\partial c}=-\sum_{i=1}^{m}\frac{x_i^2}{(a+bx_i+cx_i^2)^2}-\frac{1}{3}(d-1)\sum_{i=1}^{m}\frac{x_i^4e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]^2}$$

$$-\frac{1}{3}\frac{(n-m)d^2x_m^4e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-2}}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2}$$

$$+\frac{1}{3}\frac{(n-m)d(d-1)x_m^4e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-2}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$-\frac{1}{3}\frac{(n-m)dx_m^4 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d}},$$

$$\frac{\partial^2\mathcal{L}}{\partial b^2} = -\sum_{i=1}^{m}\frac{x_i^2}{(a+bx_i+cx_i^2)^2} - \frac{1}{4}(d-1)\sum_{i=1}^{m}\frac{x_i^4 e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]^2}$$

$$-\frac{1}{4}\frac{(n-m)d^2x_m^4 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-2}}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d}\right]^2}$$

$$+\frac{1}{4}\frac{(n-m)d(d-1)x_m^4 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-2}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d}}$$

$$-\frac{1}{4}\frac{(n-m)dx_m^4 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d}},$$

$$\frac{\partial^2\mathcal{L}}{\partial b\partial c} = -\sum_{i=1}^{m}\frac{x_i^3}{(a+bx_i+cx_i^2)^2} - \frac{1}{6}(d-1)\sum_{i=1}^{m}\frac{x_i^5 e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]^2}$$

$$-\frac{1}{6}\frac{(n-m)d^2x_m^5 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-2}}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d}\right]^2}$$

$$+\frac{1}{6}\frac{(n-m)d(d-1)x_m^5 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-2}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$-\frac{1}{6}\frac{(n-m)dx_m^5 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d},$$

$$\frac{\partial^2\mathcal{L}}{\partial c^2}=-\sum_{i=1}^{m}\frac{x_i^4}{(a+bx_i+cx_i^2)^2}-\frac{1}{9}(d-1)\sum_{i=1}^{m}\frac{x_i^6 e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}}{\left[e^{(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3)}-1\right]^2}$$

$$-\frac{1}{9}\frac{(n-m)d^2x_m^6 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{2d-2}}{\left[-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d\right]^2}$$

$$+\frac{1}{9}\frac{(n-m)d(d-1)x_m^6 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-2}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}$$

$$-\frac{1}{9}\frac{(n-m)dx_m^6 e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^{d-1}}{-1+\left[1-e^{-(ax_m+\frac{b}{2}x_m^2+\frac{c}{3}x_m^3)}\right]^d}.$$

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